PDE QUALIFYING EXAMINATION (MAY 2017)

Do not use any outside resources or collaborate with anyone to help you complete this exam. You have 2 hours to complete this exam.

Problem 1. Fix $1 \le p < \infty$, $n \ge 3$, and an integer $k > 0$ such that $n - kp > 0$. Define the operator $T : C(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n) \to C(\mathbb{R}^{n-2})$ by $Tu := u|_{\mathbb{R}^{n-2}}$, where we identify $\mathbb{R}^{n-2}$ with the set $\{x \in \mathbb{R}^n \mid x_{n-1} = x_n = 0\}$. Prove that the following claim **cannot** be true if $q \notin [\frac{p(n-2)}{n}, \frac{p(n-2)}{n-kp}]$: there exists a constant $C > 0$ depending only on k, p, q, and n such that for any $u \in C(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$,

$$\|Tu\|_{L^q(\mathbb{R}^{n-2})} \le C\|u\|_{W^{k,p}(\mathbb{R}^n)}.$$

Problem 2. Let Ω be an open, bounded set in $\mathbb{R}^n$ with smooth boundary where $n \ge 2$. Fix $\frac{n}{n-1} < p < n$ and let $\frac{1}{p} + \frac{1}{p'} = 1$, and for any $\phi \in L^q(\Omega)$ where $1 \le q \le \infty$, define F_ϕ on $W^{1,p}(\Omega) \times W^{1,p'}(\Omega)$ by

$$F_\phi(u,v) := \int_\Omega \phi(x)u(x)v(x)dx,$$

and define

$$I := \{q \in [1,\infty] \mid F_\phi(u,v) < \infty \text{ for all } (u,v) \in W^{1,p}(\Omega) \times W^{1,p'}(\Omega) \text{ and } \phi \in L^q(\Omega)\}.$$

Prove that

$$[\frac{n}{2}, \infty] \subset I.$$

Problem 3. Take the same setting as problem 2 above but with $p = \frac{n}{n-1}$. Prove that

$$(\frac{n}{2}, \infty] \subset I.$$

Problem 4. Let $\Omega = B_1(0) \subset \mathbb{R}^n$ be the open unit ball, and $\phi \in C(\mathbb{R}^n \times \Omega) \setminus C^1(\mathbb{R}^n \times \Omega)$. Prove that if u, $w \in C^2(\Omega) \cap C(\overline{\Omega})$ are such that $u \le w$ on $\partial\Omega$ and

$$-\sum_{i=1}^n (1 - e^{-|Du|})u_{ii} + \phi(Du, x) + u \le -\sum_{i=1}^n (1 - e^{-|Dw|})w_{ii} + \phi(Dw, x) + w$$

on Ω, then $u \le w$ on all of Ω.

Problem 5. Let $\Omega = (0,\pi) \times (0,\pi) \subset \mathbb{R}^2$, and we notate points $(x,y) \in \mathbb{R}^2$. Also define $Lu = -D_j(a^{ij}u_i)$, where

$$(a^{ij}(x,y)) := \begin{pmatrix} 2 & \frac{xy}{\pi^2} \\ \frac{xy}{\pi^2} & 2 \end{pmatrix}$$

and let λ_1 be the principal eigenvalue for the Dirichlet problem associated to L in the **weak** sense. Prove that

$$\frac{1}{\pi^2} \le \lambda_1 \le 12.$$